THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2020B Advanced Calculus II
Suggested Solutions for Homework 11
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1. Let $\mathbf{F} = (y\cos 2x)\mathbf{i} + (y^2 \sin 2x)\mathbf{j} + (x^2 y + z)\mathbf{k}$. Is there a vector field $\mathbf{A}$ such that $\mathbf{F} = \nabla \times \mathbf{A}$? Explain your answer.

   Solution. We test whether $\mathbf{F}$ is divergence-free (that is, $\nabla \cdot \mathbf{F} = 0$, see Theorem 9 of Section 15.8 of the textbook). We have

$$\begin{aligned}\nabla \cdot \mathbf{F} &= \frac{\partial}{\partial x}(y\cos 2x) + \frac{\partial}{\partial y}(y^2\sin 2x) + \frac{\partial}{\partial z}(x^2y+z)\\ &= -2y\sin 2x + 2y\sin 2x + 1\\ &= 1.\end{aligned}$$

   Since $\mathbf{F}$ is not divergence free, it is not the curl of another vector field $\mathbf{A}$. ◀

2. Compute the net outward flux of the vector field

$$\mathbf{F} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2+y^2+z^2)^{3/2}}$$

   across the ellipsoid $9x^2 + 4y^2 + 6z^2 = 36$.

   Solution. By the divergence theorem, we have

$$\iint\limits_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint\limits_D \nabla \cdot \mathbf{F}\, dV$$

   and so we compute the divergence

$$\begin{aligned}\nabla \cdot \mathbf{F} &= \frac{\partial}{\partial x}\left(\frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2+y^2+z^2)^{3/2}}\right) + \frac{\partial}{\partial y}\left(\frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2+y^2+z^2)^{3/2}}\right) + \frac{\partial}{\partial z}\left(\frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2+y^2+z^2)^{3/2}}\right)\\ &= \frac{-2x^2+y^2+z^2}{(x^2+y^2+z^2)^{3/2}} + \frac{x^2-2y^2+z^2}{(x^2+y^2+z^2)^{3/2}} + \frac{x^2+y^2-2z^2}{(x^2+y^2+z^2)^{3/2}}\\ &= 0.\end{aligned}$$

   Hence,

$$\iint\limits_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint\limits_{9x^2+4y^2+6z^2=36} \nabla \cdot \mathbf{F}\, dV = \iiint\limits_{9x^2+4y^2+6z^2=36} 0\, dV = 0.$$

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3. Find the flux of the vector field:

$$\mathbf{F} = (5x^3 + 12xy^2)\mathbf{i} + (y^3 + e^y \sin z)\mathbf{j} + (5z^3 + e^y \cos z)\mathbf{k},$$

through the boundary of the region D as the solid between the spheres:

$$x^2 + y^2 + z^2 = 1 \quad \text{and} \quad x^2 + y^2 + z^2 = 2.$$

Solution. We again use the divergence theorem. The divergence of $\mathbf{F}$ is

$$\begin{aligned}\nabla \cdot \mathbf{F} &= \frac{\partial}{\partial x}\left(5x^3 + 12xy^2\right) + \frac{\partial}{\partial y}\left(y^3 + e^y \sin z\right) + \frac{\partial}{\partial z}\left(5z^3 + e^y \cos z\right) \\ &= 15x^2 + 12y^2 + 3y^2 + e^y \sin z + 125z^2 - e^y \sin z \\ &= 15(x^2 + y^2 + z^2) \\ &= 15\rho^2\end{aligned}$$

where $\rho^2 = x^2 + y^2 + z^2$.

Then in spherical coordinates we have

$$\begin{aligned}\iint_S \mathbf{F} \cdot \mathbf{n} d\sigma &= \iiint_D \nabla \cdot \mathbf{F} dV \\ &= \int_0^{2\pi} \int_0^{\pi} \int_1^{\sqrt{2}} 15\rho^4 \sin\phi d\rho d\phi d\theta \\ &= \int_0^{2\pi} \int_0^{\pi} (12\sqrt{2} - 3) \sin\phi d\phi d\theta \\ &= \int_0^{2\pi} 2(12\sqrt{2} - 3) d\theta \\ &= 12\pi(4\sqrt{2} - 1).\end{aligned}$$

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4. A function $f(x, y, z)$ is said to be *harmonic* in a region D in space if it satisfies the *Laplace equation*

$$\nabla^2 f = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

throughout D.

(a) Suppose that f is harmonic throughout a bounded region D enclosed by a smooth surface S and that $\mathbf{n}$ is the chosen unit normal vector on S. Show that the integral over S of $\nabla f \cdot \mathbf{n}$, the derivative of f in the direction of $\mathbf{n}$, is zero.

(b) Show that if f is harmonic on D, then

$$\iint_S f \nabla f \cdot \mathbf{n} d\sigma = \iiint_D \|\nabla f\|^2 dV.$$

Solution. (a) By the divergence theorem, we have

$$\iint_S \nabla f \cdot \mathbf{n} d\sigma = \iiint_D \nabla \cdot (\nabla f) dV = \iiint_D \nabla^2 f dV = \iiint_D 0 dV = 0.$$

(b) By the divergence theorem and the product rule, we have

$$\begin{aligned} \iint_S f \nabla f \cdot \mathbf{n} d\sigma &= \iiint_D \nabla \cdot (f \nabla f) dV \\ &= \iiint_D \left(\nabla f \cdot \nabla f + f \nabla^2 f \right) dV \\ &= \iiint_D \left(\|\nabla f\|^2 + f \cdot 0 \right) dV \\ &= \iiint_D \|\nabla f\|^2 dV. \end{aligned}$$

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5. Suppose that f and g are scalar functions with continuous first- and second-order partial derivatives throughout a region D that is bounded by a closed piecewise smooth surface S. Show that

$$\iint_S f \nabla g \cdot \mathbf{n} d\sigma = \iiint_D \left(f \nabla^2 g + \nabla f \cdot \nabla g \right) dV.$$

The above equation is *Green's first formula.* (*Hint:* Apply the Divergence Theorem to the field $\mathbf{F} = f \nabla g$.)

Solution. Following the hint, and using product rule, we have

$$\begin{aligned} \iint_S f \nabla g \cdot \mathbf{n} d\sigma &= \iiint_D \nabla \cdot (f \nabla g) dV \\ &= \iiint_D \left(\nabla f \cdot \nabla g + f \nabla^2 g \right) dV \end{aligned}$$

as required. ◀